

# Bulk Entropy in Loop Quantum Gravity

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## ABSTRACT

In the framework of loop quantum gravity (LQG), having quantum black holes in mind, we generalize the previous boundary state counting (gr-qc/0508085) to a full bulk state counting. After a suitable gauge fixing we are able to compute the bulk entropy of a bounded region (the “black hole”) with fixed boundary. This allows us to study the relationship between the entropy and the boundary area in details and we identify the holographic regime of LQG where the leading order of the entropy scales with the area. We show that in this regime we can fine tune the factor between entropy and area without changing the Immirzi parameter.

## I. INTRODUCTION

To understand the deep structure of Loop Quantum Gravity (LQG) [1], in particular the holographic principle and quantum black holes, it is necessary to analyze in detail the entropy counting. Most of the work on black hole entropy in LQG focuses on boundary state counting, especially in the isolated horizon framework [2]. In the present work we propose to extend these considerations to the bulk entropy.

We use the framework outlined in [3]. We consider a given spin network state for the 3d geometry of the (canonical) hypersurface and focus on an arbitrary bounded region. The previous work [3] analyzed the boundary entropy of such a region assuming a totally mixed state, i.e having no knowledge about its interior. We naturally recovered the area-entropy law. In the present work, we extend this calculation to the bulk entropy. More precisely, we retain the information about the graph (underlying the quantum geometry state) inside the considered region and, under given boundary conditions, count all the distinct spin network states on that graph. Then the entropy – defined as the logarithm of the number of states – depends on the topology of the graph through its number of loops.

In the trivial topology case with no loop, we recover the previous result obtained by counting boundary states [3]. As soon as the topology becomes non-trivial the entropy diverges. Nevertheless, we identify a symmetry responsible for this divergence and get a finite entropy after suitable gauge fixing. The entropy increases with the number of loops. For a complicated enough graph topology we find that the entropy can grow arbitrarily large compared to the boundary area. However, we also identify a regime for which the entropy still scales as the area. We call it the “holographic regime”. There it turns out to be possible to arbitrarily adjust the proportionality factor between the entropy and the area (without changing the Immirzi parameter). If quantum gravity is to be a holographic theory, then the LQG dynamics should be such that projecting on physical states selects this regime.

Finally, the gauge fixing that we use seems to be related to a gauge fixing of the Hamiltonian constraint, but more investigations are required to understand that relationship.

## II. GRAPH TOPOLOGY AND STATE COUNTING

Let us start with an arbitrary spin network state on the canonical hypersurface  $\Sigma$ . Consider a connected bounded region  $\mathcal{R}$  of its graph<sup>1</sup> that includes a finite set of vertices and the edges that connect them. The boundary  $\partial\mathcal{R}$  is

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<sup>1</sup> It is usually assumed that the graph is locally finite, i.e that the valency of each vertex is finite.

the set of edges which have only one end vertex laying in  $\mathcal{R}$ . We can picture  $\mathcal{R}$  as a 3-ball and  $\partial\mathcal{R}$  as its boundary 2-sphere punctured by the boundary edges.

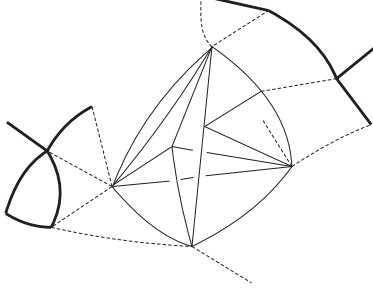


FIG. 1: The internal edges are shown as regular lines, the boundary edges are dashed, and the selected exterior edges are rendered in bold.

Let us call  $\Gamma$  the graph inside  $\mathcal{R}$  and assume that the boundary  $\partial\mathcal{R}$  is made of  $n$  edges. The spin network state carries spin labels ( $SU(2)$  representations) attached to each edge of the graph. In particular, the boundary data consists of the spin labels  $j_1, \dots, j_n$  of the  $n$  boundary edges puncturing  $\partial\mathcal{R}$ . We assume these labels fixed once and for all as defining the boundary. We further take them all equal to the lowest spin  $j_1 = \dots = j_n = 1/2$ . As it was shown in [3], this hypothesis simplifies the calculations and it is straightforward to generalize the calculations to any generic configuration. For such a choice of spins the number of boundary edges  $n$  is necessarily even, and we therefore assume in the following that the boundary is made of  $2n$  edges.

In [3] we motivated defining boundary states as intertwiners between the  $2n$  boundary representations — or equivalently as singlet states in the tensor product of the  $2n$  spin  $j_1, \dots, j_{2n}$ . Counting the dimension  $N_\partial$  of the intertwiner space<sup>2</sup> gives the (boundary) entropy  $S_\partial$  in term of the binomial coefficient  $C_{2n}^n$ :

$$S_\partial \equiv \log N_\partial = \log \frac{1}{n+1} \binom{2n}{n} \underset{n \rightarrow \infty}{\sim} 2n \log 2 - \frac{3}{2} \log n + \dots \quad (1)$$

Assuming that every  $j = 1/2$  puncture contributes a microscopic area  $a_{\frac{1}{2}} l_P^2$  in Planck units, we find that the entropy satisfies the usual proportionality law at the leading order, and has a  $-3/2$  logarithmic correction:

$$S_\partial \sim \frac{A}{l_P^2} \frac{\log 2}{a_{\frac{1}{2}}} - \frac{3}{2} \log A + \dots, \quad (2)$$

where the total boundary area is  $A = 2na_{\frac{1}{2}} l_P^2$ . In the standard LQG framework, the microscopic area is given in terms of the Immirzi parameter  $a_{\frac{1}{2}} = \gamma\sqrt{3}/2$ . Then the semi-classical relation  $S \sim A/4l_P^2$  can be used to fix the value of the Immirzi ambiguity  $\gamma$ . The isolated horizon framework presents a different though similar calculation for the entropy. It leads to a different ratio  $S/A$  and thus a different value of  $\gamma$ , but also to a different value of the logarithmic correction - usually  $\frac{1}{2}$  instead of the present  $\frac{3}{2}$ .

This boundary entropy calculation can be interpreted as computing the number of spin network states assuming that the graph  $\Gamma$  inside the region  $\mathcal{R}$  is reduced to a single vertex. Assuming that the outside observer have no access to any information about the graph inside  $\mathcal{R}$ , we have indeed coarse-grained this graph to a single point: this can be dubbed the “black point” model.

In the present work we investigate the effects of keeping a non-trivial graph  $\Gamma$  on the entropy calculation. As a spin-network wave functional is a function of the (gravitational) holonomies, it is possible to have an impression that

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<sup>2</sup> In [3], we assumed the black hole state defined by the totally mixed state  $\rho \propto \mathbb{1}$  on the intertwiner space. This is naturally the state seen by the external observer who does not have any information on the internal black hole state. It is also a static state, which does not evolve under unitary evolution, and thus corresponds to the physical set-up of a black hole. Then the entropy of this state is of course the logarithm of the intertwiner space dimension.

degrees of freedom are attached to the edges of  $\Gamma$ . However, due to the gauge invariance, degrees of freedom are truly carried by the loops of the graph.

Consider  $\Gamma$  with  $V$  vertices,  $E$  internal edges and  $E_\partial = 2n$  external legs. Then the number of independent loops is  $L = E - V + 1$ . When  $\Gamma$  is a tree,  $L = 0$ , the entropy is exactly the same as for the trivial graph with a single vertex. As soon as the graph possesses a non-trivial topology<sup>3</sup>,  $L \geq 1$ , we obtain an infinite number of states and thus of degrees of freedom. Indeed, loops of a spin network can carry arbitrary spins  $j \in \mathbb{N}/2$  independently of each other. To overcome this obstacle, we note that this divergence is associated to a further gauge invariance. This gauge invariance is generated by the action of the holonomy operators on the loops of the spin network state. To count physical degrees of freedom and to get a meaningful finite entropy, we gauge fix this action [5]. We use the simplest gauge fixing and fix the spin associated to each loop of  $\Gamma$ . It actually requires fixing the spin of only one link of each loop. Having done this, we will see in the following section that we indeed obtain a finite dimensional Hilbert space. The resulting entropy does not depend of the choice of the spins used in the gauge fixing and only depends on the size/area of the boundary  $E_\partial = 2n$  and on the complexity of the graph  $\Gamma$  described by the number of loops  $L$ .

This gauge fixing of the holonomy operators on the graph loops can be related to the gauge fixing of the Hamiltonian constraint. In the context of the topological BF theory, this would actually exactly coincide with the action of the Hamiltonian constraint<sup>4</sup>. In the case of LQG, it is more subtle and we should investigate further the validity and physical interpretation of our gauge fixing. While we postpone this for future work, we insist that in our framework this gauge fixing is natural. First, we gauge fix the holonomies on loops inside the considered region  $\mathcal{R}$ . This a priori does not affect the dynamics of the exterior of  $\mathcal{R}$ . On the other hand, if we did not gauge fix them we would be over-counting the number of states (seen by an external observer). What would be non-trivial is to gauge fix the action of holonomies on loops crossing the boundary/horizon  $\partial\mathcal{R}$ . Second, we do check that the number of degrees of freedom does not depend on the gauge fixing and that the action of the holonomies creates isomorphic copies of the same “physical” Hilbert space. In short, we are disregarding the internal excitations that do not couple to the exterior of the region and that would produce an overcounting in the entropy ascribed to  $\mathcal{R}$  by an external observer.

In the following sections we study in detail the one-loop case and then show that the calculations can be straightforwardly generalized to an arbitrary graph topology. Finally we discuss the different regimes of entropy corresponding to the different scaling of the number of loops  $L$  with the size of the boundary  $n$ .

### III. THE ONE-LOOP CASE

Let us start by reviewing the black-point case,  $L = 0$ . The Hilbert space of spin networks of a tree is isomorphic to the space of intertwiners between the boundary representations [3]. Its dimension is given by:

$$\dim \mathcal{H}_0 = \int_{\text{SU}(2)} dg \chi_{\frac{1}{2}}(g)^{2n} = \frac{1}{n+1} \binom{2n}{n}, \quad (3)$$

where  $dg$  is the normalized Haar measure on the  $\text{SU}(2)$  Lie group and  $\chi_{1/2}(g)$  is the character in the fundamental spin-1/2 representation. This formula can be interpreted in terms of a random walk with a mirror in origin. The binomial coefficients  $C_{2n}^n$  give the number of returns to the origin for the usual random walk, while the mirror introduces the factor  $1/(n+1)$ .

Consider now the one-loop diagram on the left of Fig. 2. It is a loop to which  $2n$  legs are attached, so all the vertices are 3-valent<sup>5</sup>. Pick an arbitrary link  $e_0$  and fix the spin which it carries,  $j_{e_0} = j$ . This obviously fixes the action of the holonomy operator on the loop: acting with a holonomy carrying a spin  $J$  on this state would change the spin- $j$  representation into the tensor product representation  $j \otimes J$ .

<sup>3</sup> Let us point out that the graph topology does not a priori have any relation to the topology of the spatial hypersurface. There are two points of view about the manifold topology. It is either assumed right at the start and we consider embedded graphs. Otherwise we consider abstract graphs, containing only combinatorial and algebraic data, and the hypersurface topology is an emerging semi-classical notion. We favor the latter point of view.

<sup>4</sup> The action of the holonomy operators in the BF theory generates the translational symmetry on the B field. It needs to be dealt with in order to get the physical degrees of freedom of the theory

<sup>5</sup> An arbitrary one-loop graph is equivalent to this form due to the  $\text{SU}(2)$  gauge invariance at every vertex (see e.g. [5]).

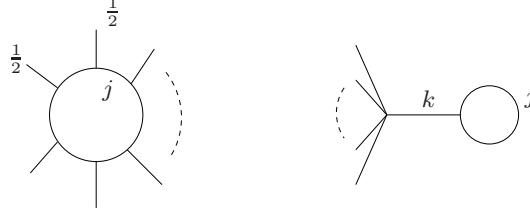


FIG. 2: One-loop spin networks with the  $2n$  boundary links: the internal loop carries the spin  $j$ .

Assume that  $j \geq n/2$ . Moving from the initial link  $e_0$  to the next link  $e_1$ , we see that  $e_1$  can carry either the spin  $j - 1/2$  or  $j + 1/2$  since the external leg between  $e_0$  and  $e_1$  injects a spin  $1/2$  into the diagram. Going on one finally arrives back to the initial link  $e_0$  and the initial representation  $j$ . Therefore, the number of possible spin labeling  $j, j \pm \frac{1}{2}, \dots, j \pm \frac{1}{2}$  is exactly the number of returns to the origin of a random walk after  $2n$  iterations:

$$\dim \mathcal{H}_1 = \binom{2n}{n}. \quad (4)$$

We get a finite result which is different from the tree case. Moreover, it does not depend on the chosen spin  $j$ . This Hilbert space leads to the following asymptotic behavior of the entropy:

$$S_1 \underset{n \rightarrow \infty}{\sim} 2n \log 2 - \frac{1}{2} \log n + \dots \quad (5)$$

It does not affect the leading order proportional to the area but only the logarithmic correction.

There is nevertheless a subtlety: the assumption that the spin  $j$  is large enough compared to the boundary size  $n$ . Indeed if  $j$  is smaller than  $n/2$ , there exists the possibility that the random walk along the loop will ascribe a spin 0 to a link. In such a case, we can not move down to a  $-1/2$  spin but can only go back up to  $+1/2$ . Then we get a smaller Hilbert space. We discard this case because we do not consider it as a true one-loop graph anymore. Indeed, if a link is labeled with a spin 0, it is just as if that link did not exist since the corresponding spin network wave function would not depend on the holonomy on that link. Then if we remove that link from the graph, we end up with a tree again. Therefore, we interpret this situation when  $j \leq n/2$  as describing a superposition of a 0-loop and 1-loop graphs. However, we can still compute exactly the dimension of the Hilbert space. For this purpose, it is more convenient to use the another one-loop diagram that is shown on the right of Fig. 2.

We now look at the one-loop diagram with two vertices: all the boundary links merge into a single link to which the loop is then attached. Assume that the loop still carries a fixed spin  $j$  and we call  $k$  the spin carried by the intermediate link. On one side, we have an intertwiner between the  $2n$  spin  $1/2$  and the spin  $k$ . As long as  $k \leq n$ , the dimension of this intertwiner space is [3]:

$$d_k^{(n)} = \binom{2n}{n+k} - \binom{2n}{n+k+1} = \frac{2k+1}{n+k+1} \binom{2n}{n+k}. \quad (6)$$

On the other side, we have a unique 3-valent intertwiner between two representations  $j$  and the same representation  $k$ . The corresponding intertwiner space is of dimension one as long as  $k \leq 2j$ . The total number of spin network states amounts to summing over all possible  $k$ 's from 0 to the maximal allowed spin  $k_{max} \equiv \min(n, 2j)$ :

$$N_1^{(j)} = \sum_{k=0}^{k_{max}} d_k^{(n)} = \binom{2n}{n} - \binom{2n}{n+k_{max}+1}. \quad (7)$$

When  $j$  is larger than  $n/2$ , we recover the previous result with  $N_1^{(j)} = N_1 = \binom{2n}{n}$ . At the other end of the spectrum, when  $j$  vanishes, we recover the 0-loop case with  $N_1^{(j)} = N_0 = \binom{2n}{n} - \binom{2n}{n+1}$ . Finally, we see that the values of the spin  $j$  carried by the loop between 0 and  $n/2$  interpolate between the tree case and the 1-loop case. This allows us to interpret this intermediate situation as a superposition of the 0-loop and 1-loop cases, and moreover to identify the true 1-loop case as the  $j \geq n/2$  regime when the entropy does not depend anymore on the specific value of  $j$ .

To conclude this section, we showed that considering a one-loop graph over a tree increases the entropy. After carefully gauge fixing, we obtain a finite entropy which differs from the 0-loop case only by the logarithm correction,  $-1/2$  instead of  $-3/2$ .

In the following sections, we generalize this analysis to an arbitrary number of loops. We will show that if the number of loops is fixed it will only affect the logarithm correction while if we allow the number of loops to scale with  $n$  it may well also affect the leading order.

#### IV. THE TWO-LOOP CASE

We now move to the entropy counting in the two-loop case. We work with the graph shown on fig.3 and we fix the spins along the two loops to the same value  $j$  for the sake of simplicity. We further assume as previously that  $j \geq n/2$  so that we consider a pure two-loop graph and not a superposition with a 0-loop or 1-loop configuration. The dimension of the Hilbert space of spin networks on  $\Gamma$  is given by the following integral:

$$N_2 = \int dg \chi_{\frac{1}{2}}(g)^{2n} \chi_j(g)^4.$$

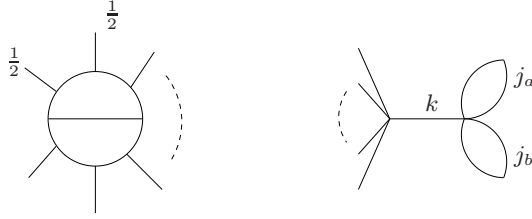


FIG. 3: Two-loop graphs with boundary links

For computational purpose, it is convenient to express as a sum over the intermediate spin label  $k$ . It is straightforward to obtain

$$N_2 = \sum_{k=0}^n d_k^{(n)} \left[ (2j+1)(2k+1) - \frac{3}{2}k(k+1) \right]. \quad (8)$$

The two terms can be computed exactly<sup>6</sup>:

$$\sum_{k=0}^n (2k+1) d_k^{(n)} = 2^{2n}, \quad \sum_{k=0}^n k(k+1) d_k^{(n)} = n \binom{2n}{n} \sim \sqrt{n} 2^{2n}. \quad (9)$$

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<sup>6</sup> A useful identity for the degeneracy coefficients is

$$\sum_{k=0}^n d_k^{(n)} \sin(2k+1)\theta = 2^{2n} \sin \theta \cos^{2n} \theta,$$

which is derived by computing the trace of  $SU(2)$  group elements in the representation  $(\frac{1}{2})^{\otimes 2n}$ . Differentiating this equation, we obtain the following formula for the polynomial averages over the  $d_k^{(n)}$  distribution, for  $l \in \mathbb{N}$ :

$$\sum_{k=0}^n (2k+1)^{2l+1} d_k^{(n)} = (-1)^l 2^{2n} \partial_\theta [\sin \theta \cos^{2n} \theta]_{\theta=0}.$$

As we see we have a residual  $j$ -dependent term, which actually diverges as  $j$  grows to infinity. Neglecting<sup>7</sup> the extra-term  $k(k+1)$ , the  $j$ -dependence can be factorized. Therefore, we interpret this as a symmetry that we haven't gauge fixed yet. Indeed we have gauge fixed the action of the holonomy operator along the loops  $a$  and  $b$  by requiring  $j_a = j_b = j$ , but the action of the holonomy along the loop  $a \cup b$  is not yet fixed. This leaves a freedom in the intertwiner which leads to that  $(2j+1)$  factor, resulting in a over-counting in the Hilbert space dimension.

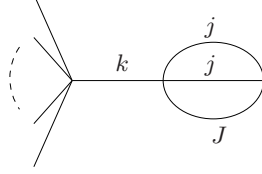


FIG. 4: Fully gauge-fixed two-loop spin network: we fix the spins carried by the three edges on the loops.

This can easily be seen considering the two-loop diagram as shown in Fig. 4.  $J$  can still vary from 0 to  $(2j+1)$ . This is the freedom that requires gauge-fixing. It is possible to compute the intertwiner dimension for different values of  $J$ . It is possible to compute the intertwiner dimension for different values of  $J$ . For instance, we get:

$$N_2^{(J=j)} = \sum_{k=0}^n (2k+1) d_k^{(n)}, \quad N_2^{(J=2j)} = \sum_{k=0}^n (k+1) d_k^{(n)}.$$

Both these examples, have the same behavior up a factor with a leading order given by  $\sum_k k d_k^{(n)}$ . They give the same entropy in the asymptotic limit  $n \rightarrow +\infty$  (same leading order and logarithmic correction):

$$S_2 \sim \log 2^{2n} \sim 2n \log 2 + \dots \quad (10)$$

with a vanishing logarithmic correction (they are of course sublog corrections). We see that adding one loop from the one-loop case corresponds to implementing a  $+\frac{1}{2}$  factor in the logarithmic correction without affecting the leading order proportional to the area. We generalize this statement to arbitrary number of loops in the following section.

## V. GAUGE FIXING AND GENERIC BULK ENTROPY

### A. Informal Arguments

For a generic graph with  $L$  loops, following the one-loop and two-loop cases, it is reasonable to expect the gauge-fixed dimension at leading order to scale as

$$N_L \sim \sum_{k=0}^n k^{L-1} d_k^{(n)}. \quad (11)$$

Assuming that  $L$  is fixed, the asymptotics in the large  $n$  limit only changes the logarithmic correction of the entropy:

$$S_L \equiv \log N_L \sim 2n \log 2 + \left( \frac{L}{2} - 1 \right) \log n + \dots \quad (12)$$

Using the same techniques as in [3], it is straightforward to approximate the sum over  $k$  by an integral and then compute the asymptotics of  $N_L$  by a saddle point approximation. Then only the logarithmic correction changes by a

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<sup>7</sup> The second term in  $k(k+1)$  is indeed neglected if  $j$  goes to infinity faster than  $\sqrt{n}$ . This condition is automatically satisfied since we assume that  $j \geq n/2$ .

factor  $+\frac{1}{2}$  with each loop because the distribution  $d_k^{(n)}$  looks like a Gaussian peaked at  $k = 0$  with a width scaling as  $\sqrt{n}$ .

More precisely, denoting  $x \equiv k/n \in [0, 1]$ , we can approximate the dimensions  $d_k^{(n)}$  for large  $n$  using the Stirling formula:

$$d_k^{(n)} \sim \frac{2}{\sqrt{\pi}} \frac{2^{2n}}{\sqrt{n}} \frac{x}{(1+x)\sqrt{1-x^2}} e^{-n\varphi(x)}, \quad (13)$$

with the exponent  $\varphi(x)$  given by:

$$\varphi(x) = (1+x)\log(1+x) + (1-x)\log(1-x). \quad (14)$$

Therefore, the full Hilbert space dimension can be approximated by an integral:

$$N_L \sim \frac{2}{\sqrt{\pi}} 2^{2n} n^{L-\frac{1}{2}} \int_0^1 dx \frac{x^L}{(1+x)\sqrt{1-x^2}} e^{-n\varphi(x)}.$$

The exponent  $\varphi(x)$  is always positive for  $x \in [0, 1]$ . It has a unique fixed point at  $x = 0$ . For large  $n$ , we can use the Gaussian approximation,  $\varphi(x) = x^2 + \mathcal{O}(x^3)$ . Then we need to evaluate  $\int_0^\infty dx x^L \exp(-nx^2)$ , which scales as  $n^{-(L+1)/2}$ . Finally, this leads to the asymptotics for the entropy given above in (12).

So far we have kept the number of loops  $L$  fixed as the boundary area  $n$  was taken to infinity. This number does not affect the leading order of the entropy, but only changes the pre-factor of the logarithmic correction. However, we see that if we allow the graph complexity to grow with the boundary size, we are able to change the leading order behavior of the entropy and change the proportionality factor between the entropy  $S$  and the area  $n$ .

Below we make this argument more precise and compute the dimension and entropy exactly for a specific choice of gauge-fixing. We find a missing factor  $1/(L-1)!$  in the guessed dimension (11). This changes the asymptotical behavior when  $L$  is allowed to be arbitrarily large. Then we keep the asymptotics described above for fixed  $L$  while allowing the loop number  $L$  to scale as  $n$  changes the leading order factor<sup>8</sup> between the  $S_L$  and  $n$ .

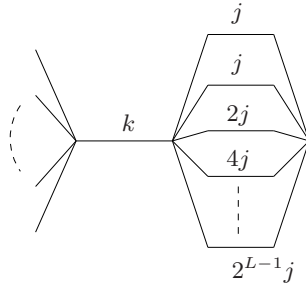


FIG. 5: Gauge-fixed spin networks with  $L$  loops: the  $L+1$  edges of  $\Gamma$  carry the labels  $j, j, 2j, \dots, 2^{L-1}j$ .

<sup>8</sup> Assuming that the number of loops goes as  $L = \alpha n$  with a fixed ratio  $\alpha > 0$ , it is straightforward to extract the asymptotics of the entropy:

$$S_\alpha \equiv \log \left[ \frac{1}{(L-1)!} \sum_{k=0}^n k^{L-1} d_k^{(n)} \right] \sim \lambda n - \frac{1}{2} \log n + \dots,$$

where the precise value of  $\lambda > 0$  depends on  $\alpha$  but is generically different from  $2 \log 2$ . The key to this calculation is that the weight  $k^{L-1}$  now contributes to the exponent which get modified to  $\tilde{\varphi}(x) = \varphi(x) - \alpha \log x$ . The fixed point  $x_0$  of  $\tilde{\varphi}$  is not  $x = 0$  anymore. In particular  $\tilde{\varphi}(x_0) \neq 0$  and this leads to a term proportional to  $n$  in the Gaussian approximation.

## B. The Entropy Formula

Let us now consider the  $L$ -loop graph with three vertices  $A, B, C$  as shown on Fig 5. The  $2n$  boundary links combine into intermediate spin  $k$  at the vertex  $A$ . On the other side, the  $L$  loops combine at the vertex  $B$  to that same intermediate link. Finally the intertwiner at the  $C$  node closes the graph and describes how the  $L$  loops are coupled to each other.

Our “gauge fixing”, or more precisely, the choice of a sector of the spin network Hilbert space, is achieved by fixing the spin labels on the  $L+1$  edges defining the  $L$  loops. We label them sequentially by representations  $j, j, 2j, 4j, 8j, \dots, 2^{L-1}j$ . As long as  $j$  is assumed larger than  $n/2$  as before, this leads to an entropy which does not depend on the gauge-fixing representation  $j$  but only on the number of loops  $L$  and the number of boundary edges  $n$ .

We need to underline that we are not only fixing the action of the holonomy operator along the  $L$  loops. Indeed, our specific choice of representation labels is not only convenient for computational purposes but also it fully fixes the unique intertwiner at the node  $C$ . This amounts to fixing  $L-2$  representation labels within the node  $C$ . This means that we are truly fixing  $(L+1) + (L-2) = 2L-1$  quantum numbers, and not only  $L$  as we first expected. We are in fact counting the number of possible intertwiners at the vertices  $A$  and  $B$ . At a naïve level, the entropy that we compute counts only the different ways that the internal loops couple to the external links while we disregard the number of ways that the loops couple to each other by fixing the intertwiner at the internal node  $C$ . At a mathematical level, if one fixes less representation labels than we do (less than  $2L-1$ ), one gets an entropy which depends on  $j$  and diverges as  $j$  grows large. This is a symptom of insufficient gauge-fixing. Nevertheless, there remains the open issue of understanding the physical interpretation and relevance of such a gauge-fixing. In particular, what is the precise symmetry that we are gauge-fixing? We leave this question for future investigation. However, even without such an interpretation in term of symmetry and gauge-fixing, we still have identified sectors of the spin network Hilbert space with different asymptotic behaviors of the entropy of the considered region which only depend on the number of loops of the graph (supporting the spin network states) inside that region.

To compute the entropy, we need to find the number of intertwiners at the vertices  $A$  and  $B$ . At the vertex  $A$  we have the same degeneracy  $d_k^{(n)}$  as previously. As for the vertex  $B$ , having  $L$  loops leads to the following degeneracies for the spin- $k$  subspaces as long as  $k \leq 2^{L-1}j$  for  $L \geq 1$ :

$${}_j d_k^{(L)} = \binom{k+L-1}{k}. \quad (15)$$

This formula is straightforward to prove by induction using the identity  $\sum_{k=0}^K \binom{k+L-1}{k} = \binom{K+L}{K}$ . As a result, the dimension of the spin network space inside  $\mathcal{R}$  is

$$N_L = \sum_{k=0}^n d_k^{(n)} {}_j d_k^{(L)}, \quad (16)$$

where we have assumed  $n \leq 2j$ . For  $L \geq 2$  this can be slightly simplified<sup>9</sup>:

$$N_L = \sum_{k=0}^n d_k^{(n)} \binom{k+L-1}{k} = \sum_{k=0}^n \binom{2n}{n+k} \binom{k+L-2}{k}, \quad (17)$$

For a small number of loops<sup>10</sup>, we easily get the exact results for  $N_L$ , using properties of the binomial coefficients [6]. For  $L=1$ , we recover the previous result  $N_1 = \sum_k d_k^{(n)} = \binom{2n}{n}$ , with an asymptotic expression for the entropy  $S_1 \sim 2n \log 2 - \frac{1}{2} \log n$ . For  $L=2$  one obtains

$$N_2 = \sum_{k=0}^n \binom{2n}{n+k} = 2^{2n-1} + \frac{1}{2} \binom{2n}{n}, \quad (19)$$

---

<sup>9</sup> This sum has an explicit closed form in terms of hypergeometric functions,

$$N_L = \frac{(2n)!}{(n!)^2} {}_2F_1(-n, L-1, 1+n; -1). \quad (18)$$



with the entropy  $S_2$  having no logarithmic correction for large  $n$  (of course, it still contains sub-logarithmic corrections). For  $L = 3$ , we can also compute:

$$N_3 = \sum_{k=0}^n \binom{2n}{n+k} (k+1) = \frac{n+1}{2} \binom{2n}{n} + 2^{2n-1}, \quad (20)$$

with a logarithmic correction being  $+\frac{1}{2}$ .

### C. The Entropy Asymptotics

In the analysis of the entropy asymptotics, we distinguish between three cases. First, if the number of loops  $L$  is held fixed (or more generally  $L$  stays negligible compared to  $n$ ), we recover the results given above: the leading order  $2n \log 2$  does not change, while the non-trivial topology of the graph affects the logarithmic correction which becomes  $(L/2 - 1) \log n$ .

In the second case  $L$  is scaling as  $n$ : the entropy depends on the limit of the ratio  $\alpha \equiv L/n$  as  $n$  goes to infinity. In the third case  $L$  grows very large compared to  $n$ : if the ratio  $L/n$  is unbounded, the leading order of the entropy drastically changes and will not scale proportionally to  $n$  anymore.

The asymptotic expression for all the above cases can be derived from a saddle point approximation of the sum  $N_L = \sum_k \binom{2n}{n+k} \binom{k+L-2}{k}$  as shown in details in appendix. It also allows to obtain the sub-leading terms with an excellent precision.

When  $L \ll n$  we get a compact expression,

$$S_L = \log N_L, \quad N_L \sim \frac{2^{2n-L-3} n^{L/2-1}}{\Gamma(L/2)}, \quad (21)$$

where  $\Gamma(z) = (z-1)!$ . This still gives a good estimate of the sub-leading terms: for example, when  $L = 7$  and  $n = 10000$ , the difference between the exact and asymptotic values is  $\Delta S = 0.354794$ , while  $S \approx 1.39 \times 10^4$ .

When  $L = \alpha n$  with a fixed ratio  $\alpha$ , we can still apply the same saddle point technique. Using Stirling formula, we approximate the dimension  $N_L$  by the integral expression,

$$N_L \sim \frac{2^{2n} n^{L-\frac{3}{2}}}{\sqrt{\pi}(L-2)! e^{L-2}} \int_0^1 dx \frac{1}{\sqrt{x}\sqrt{1-x^2}(x+\alpha)^{\frac{3}{2}}} e^{-n\tilde{\varphi}(x)}, \quad (22)$$

with the new  $\alpha$ -dependent exponent,

$$\tilde{\varphi}(x) = \varphi(x) + x \log x - (x+\alpha) \log(x+\alpha). \quad (23)$$

As shown in details in appendix, the Gaussian approximation controls the asymptotic behavior of the entropy:

$$S_{L=\alpha n} \sim \lambda n - \frac{1}{2} \log n + \dots \quad (24)$$

The important new result is that the leading order factor  $\lambda \sim S/n$  depends on  $\alpha$  and is not fixed to the standard factor  $2 \log 2$  anymore. We show on Fig. 6 a plot with Maple numerics for the entropy  $S_L(n)$  for  $L = 0$  (as a reference) and  $\alpha = 1, 2$ . While the leading behavior is always linear, the slope  $\lambda \equiv S/n$  clearly depends on the value of the ratio  $\alpha = L/n$ .

The final case is when the complexity of the graph  $L$  grows much faster than the boundary size  $n$ . In this case, the entropy grows faster than linearly compared to the boundary area:

$$S_{L \gg n} \sim n(\log L - \log n) + n - \frac{1}{2} \log n + \dots \quad (25)$$

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<sup>10</sup> To reach higher values of  $L$ , the number of states  $N_L$  can be calculated by induction with the help of the following recurrence relation:

$$\sum_{k=0}^n k^m \binom{2n}{n+k} = \sum_k k^{m-2} [n^2 - (n^2 - k^2)] \binom{2n}{n+k} = n^2 \sum_k k^{m-2} \binom{2n}{n+k} - 2n(2n-1) \sum_k k^{m-2} \binom{2n-2}{n-1+k}.$$

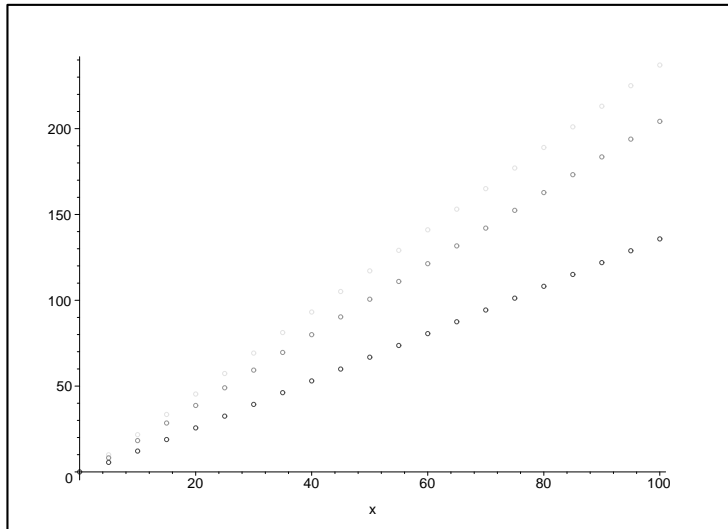


FIG. 6: Plot of the entropy  $S_L(n)$  for  $L = 0$ ,  $L = n$  and  $L = 2n$  for the number of punctures  $n$  running from 0 to 100. The leading behavior is linear in  $n$  but the slope depends on the scaling  $L/n$ . The numerics give  $\lambda \sim 1.386$  for the boundary entropy  $L = 0$ , while we get  $\lambda \sim 2.074$  for  $L = n$  and  $\lambda \sim 2.400$  for  $L = 2n$ . These numerical values fit the analytical result (up to two decimals) derived in appendix.

## VI. THE REGIMES OF BULK ENTROPY

After a suitable gauge fixing, we have managed to compute the finite bulk entropy counting the number of spin network states supported by a fixed graph  $\Gamma$  inside the considered bounded region  $\mathcal{R}$  with fixed boundary conditions. The entropy  $S$  only depends on the number of links  $n$  puncturing the boundary  $\partial\mathcal{R}$  and the number of loops  $L$  of  $\Gamma$  which quantifies the complexity of the graph. Computing the asymptotics of  $S_L$ , we distinguish three regimes.

- **The logarithmic regime :**

The number of loops  $L$  is held fixed while  $n$  is taken to infinity. The leading order of  $S_L$  is linear in  $n$  and is exactly the same as for the boundary entropy [3]. The complexity of the graph only affects the factor in front of the logarithmic correction  $\log n$ , which increases with  $L$ . Therefore, if a (quantum) black hole belongs to this regime, fixing the smallest spin  $j = \frac{1}{2}$  fixes the Immirzi parameter.

- **The holographic regime :**

The number of loops  $L$  scales proportionally to the horizon area  $n$ . The entropy still grows linearly in  $n$  in the leading order but the proportionality factor changes and depends on the ratio  $\alpha = L/n$ . This is a sector of the spin network space where the entropy still scales with the boundary area and not faster. If the full LQG theory is to be (strongly) holographic, then the dynamics should restrict the physical states to this regime: the graph complexity can not grow faster than the boundary size. On the other hand, even if the LQG theory is not restricted to this regime, a quantum black hole state should lay in this regime if we want to respect the semi-classical area-entropy law. Then since the factor  $S/n$  depends on the ratio  $\alpha$ , we can adjust this new parameter  $\alpha$  so to get  $S \sim A/4$  without changing neither the Immirzi parameter nor the area quantum  $a_{\frac{1}{2}}$ . In this scenario, the black hole entropy calculation does not fix the Immirzi parameter. From another perspective, we can say that the factor  $\alpha$  renormalizes the Immirzi parameter  $\gamma$ . Finally we need the dynamics to select the physical sector of the spin network space and provide us with the “right” value of  $\alpha$ . Of course, this scenario only holds if we use the bulk entropy and do not restrict ourselves to work with the boundary entropy.

- **The non-linear regime :**

If the number of loops grows much faster than  $n$ , i.e  $L/n \rightarrow \infty$ , then the entropy is free to grow non-linearly with  $n$ . Since the leading order is  $n \log L$ , we can, e. g., get an entropy scaling with the “volume”  $n^{3/2}$  for an

exponential growth of the type  $L \sim 2\sqrt{n}$ . Of course, one should keep in mind that the volume of  $\mathcal{R}$  actually depends on the bulk state and does not always scale as  $n^{3/2}$ . It could well be larger (or smaller) if the space is tightly curved.

## Conclusions

We explored the notion of “bulk entropy” in the framework of loop quantum gravity, in order to generalize the calculations of boundary entropy of [3]. Our aim was to compute the number of (spin network) states describing the quantum geometry of a bounded region of space, living on a fixed graph and with fixed boundary conditions. This bulk entropy is obviously related to the complexity of the graph and was found to be related to the number of loops  $L$  of that graph. The straightforward calculation then gives at first an infinite entropy. Nevertheless, after fixing a certain number of labels of the spin network states, we were able to extract meaningful finite results. This led to the identification of different sectors of the spin network Hilbert space depending on  $L$  with different asymptotical behavior of the entropy.

We can a priori have an entropy growing as large as we want if we choose a number of loops  $L$  large enough. Nevertheless, two sectors are particularly interesting. If the graph complexity is fixed while the boundary area grows, then the bulk entropy  $S$  has the same leading order than the boundary entropy but the logarithmic correction to the entropy changes and increases with  $L$ . On the other hand, if  $L$  grows linearly with the boundary area  $A$ , then the bulk entropy also grows linearly with the area but the ratio  $S/A$  depends explicitly on the ratio  $L/A$ . We call this sector the “holographic regime”.

If we want black holes to satisfy the area-entropy law, the quantum black hole state must necessarily be in the holographic regime. However, this regime is generic enough to allow for a fine-tuning of the ratio  $S/A$  to  $1/4$  without changing the Immirzi parameter (i.e the value of the minimal quantum of area). Indeed, we can always change the area-complexity ratio  $L/A$  to adjust the value of  $S/A$ . Our point of view is that it is the LQG dynamic that is supposed to select the physical regime and value of  $L/A$ .

There are two important issues to address in this scenario. First, should the considered entropy for the space region be its boundary entropy or its bulk entropy? We see a priori no reason why the degrees of freedom inside the region would not couple to the space outside. Then the bulk degrees of freedom would affect the black hole evaporation. However, the case of a black hole might (should?) be drastically different from a generic region of space. This is a question that needs to be addressed by the LQG dynamics. The second issue is directly related to our approach. Our entropy calculation relies on a partial fixing of the spin network labels. We discussed that it could be interpreted as a gauge fixing of the Hamiltonian constraint, but this needs to be worked out in details. Thus, the limits of validity of the present work depend on understanding the legitimacy of this gauge fixing. However, even if it does not turn out to be interpreted as the gauge fixing of a certain symmetry, we have nevertheless identified the different sectors of the spin network kinematical space with different bulk entropy behavior, but we still need to provide these different sectors with a proper physical interpretation.

## APPENDIX A: COMPUTING THE ASYMPTOTICS OF THE ENTROPY

The asymptotic expressions for  $n \rightarrow \infty$  and the different regimes of  $L = L(n)$  are based on the use of Stirling formula, replacement of the sum by the (Euler-MacLaurent) integral and the saddle point (Gaussian) approximation. It is convenient for the analysis to introduce a shifted loop number  $l = L - 2$  and consider the ratio  $\alpha \equiv l/n$ .

The first factor in the sum of Eq. (17) takes the form

$$\binom{2n}{n+k} \sim \frac{2^{2n}}{\sqrt{\pi n}} f(k/n) e^{-n\varphi(k/n)}, \quad (\text{A1})$$

where

$$f(x) = \frac{1}{\sqrt{1-x^2}}, \quad \varphi(x) = (1+x) \log(1+x) + (1-x) \log(1-x). \quad (\text{A2})$$

The second factor becomes

$$\binom{k+l}{k} \sim \frac{1}{e^l l!} \sqrt{\frac{(k+l)}{k}} \frac{(k+l)^{l+k}}{k^k} = \frac{n^l}{e^l l!} g(k/n, l/n) e^{-n\psi(k/n, l/n)}, \quad (\text{A3})$$

where

$$g(x, \alpha) = \sqrt{\frac{(x+\alpha)}{x}}, \quad \psi(x, \alpha) = x \log x - (x+\alpha) \log(x+\alpha). \quad (\text{A4})$$

The sum of Eq. (17) is replaced by the integral

$$N_l \sim \frac{2^{2n}}{\sqrt{\pi n}} \frac{n^{l+1}}{e^l l!} \int_0^1 dx f(x) g(x, \alpha) e^{-n\tilde{\varphi}(x, \alpha)}, \quad (\text{A5})$$

with  $\tilde{\varphi} = \varphi(x) + \psi(x, \alpha)$ . The saddle point approximation requires the knowledge of the derivatives of the exponent:

$$\partial_x \tilde{\varphi} = \log \frac{x(1+x)}{(x+\alpha)(1-x)}, \quad \partial_x^2 \tilde{\varphi} = \frac{\alpha + 2\alpha x + (2-\alpha)x^2}{(x+\alpha)(x-x^3)}. \quad (\text{A6})$$

The unique fixed point,  $\partial \tilde{\varphi}(x_0) = 0$ , in the interval  $[0, 1]$  is

$$x_0 \equiv \frac{1}{4}(\sqrt{\alpha^2 + 8\alpha} - \alpha), \quad (\text{A7})$$

and it is easy to check that  $\partial^2 \tilde{\varphi}$  is always positive on  $[0, 1]$ .

In particular, for  $0 < \alpha < \infty$  the minimum of  $\tilde{\varphi}$  is inside the interval and we use the Gaussian approximation

$$\int_0^1 dx y(x) e^{-n\tilde{\varphi}(x)} \sim y(x_0) \sqrt{\frac{2\pi}{n\partial^2 \tilde{\varphi}(x_0)}} e^{-n\tilde{\varphi}(x_0)},$$

that gives the following asymptotics for the entropy:

$$S(l, n) = \log N_l, \quad N_l \sim \frac{2^{2n+1/2} n^l}{e^l l!} f(x_0) g(x_0, \alpha) \frac{e^{-n\tilde{\varphi}(x_0, \alpha)}}{\sqrt{\partial^2 \tilde{\varphi}(x_0, \alpha)}}. \quad (\text{A8})$$

Hence the leading order term of the entropy

$$S(l, n) \sim \lambda n - \frac{1}{2} \log n, \quad (\text{A9})$$

is given in terms of  $\alpha$ :

$$\begin{aligned} \lambda &\equiv 2 \log 2 - \tilde{\varphi}(x_0) - \alpha \log \alpha, \\ &= 5 \log 2 - \alpha \log 4\alpha + \alpha \log[3\alpha + \sqrt{\alpha(\alpha+8)}] - \log[8 - \alpha(4 + \alpha - \sqrt{\alpha(\alpha+8)})], \end{aligned} \quad (\text{A10})$$

while the subleading terms have a more cumbersome appearance.

As a matter of fact, this approximation is in an excellent agreement with the exact results: e. g.,  $S(l=10, n=10000) \approx 1.39 \times 10^4$  and the error is  $\Delta S \approx -0.0077$ , increasing only to  $\Delta S(l=10, n=40000) \approx -0.0080$ . At the other extreme,  $S(l=1.5 \times 10^5, n=1000) \approx 6022.7$  with  $\Delta S \approx -8.3 \times 10^{-5}$ .

When  $\alpha$  runs from 0 to  $\infty$ , the fixed point  $x_0$  varies from 0 to 1. In the special case  $\alpha = 1$  when  $l = n$ ,

$$x_0 = \frac{1}{2}, \quad \tilde{\varphi}(x_0) = -\log 2, \quad \partial^2 \tilde{\varphi}(x_0) = 4,$$

and we get simple asymptotics:

$$S(l=n) \sim 3n \log 2 - \frac{1}{2} \log n - \frac{1}{2} \log \pi. \quad (\text{A11})$$

In this case, we can actually give a faster proof of the asymptotics:

$$N_{l=n} = \sum_{k=0}^n \binom{2n}{n+k} \binom{n+k}{k} = \sum_k \frac{(2n)!}{(n!)^2} \binom{n}{k} = \frac{(2n)!}{(n!)^2} 2^n.$$

In the limit regime  $l \ll n$ , we have

$$x_0 \underset{\alpha \rightarrow 0}{\sim} \sqrt{\frac{\alpha}{2}}, \quad \tilde{\varphi}(x_0) \sim \frac{1}{2}\alpha(\log 2 - 1 - \log \alpha), \quad \partial^2 \tilde{\varphi}(x_0) \sim 4 - \sqrt{2\alpha} + 5\alpha/2,$$

which improves the estimate of Eq. (21) to

$$\begin{aligned} S(l \ll n) = & 2n \log 2 + \frac{l}{2} \log n - \frac{l}{2} \log l + (1 - \log 2) \frac{l}{2} - \frac{1}{2} \log l - \frac{1}{2} \log(4\pi) \\ & - \frac{1}{12l} + \frac{1}{\sqrt{2}} l \sqrt{\frac{l}{n}} + \frac{5}{4\sqrt{2}} \sqrt{\frac{l}{n}} - \frac{l^2}{4n} - \frac{9}{32} \frac{l}{n} + \mathcal{O}\left(\frac{1}{n^2}\right). \end{aligned} \quad (\text{A12})$$

Finally, in the case that  $l \gg n$ , we have

$$x_0 \underset{\alpha \rightarrow \infty}{\sim} 1 - \frac{2}{\alpha}, \quad \tilde{\varphi}(x_0) \sim -(\alpha + 1) \log \alpha + (2 \log 2 - 1) - \frac{5}{\alpha}, \quad \partial^2 \tilde{\varphi}(x_0) \sim \frac{\alpha}{2} + \frac{7}{2} - \frac{1}{2\alpha},$$

and Eq. (A8) yields a simple expression for the entropy,

$$S(l \gg n) \sim n \log l - (n + \frac{1}{2}) \log n + n - \frac{1}{2} \log 2\pi + (6n - 1)/12l + \mathcal{O}(n^2/l^2). \quad (\text{A13})$$

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